String Spreading on Black Hole Horizon

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Abstract

The phenomenon of string spreading on the black hole horizon, as originally discussed by Susskind, is considered in the *exact* curved Schwarzschild background. We consider an oscillating string encircling the black hole and contracting towards the horizon. We then compute the angular and radial spreading of the string, as seen by a static observer at spatial infinity using fixed finite resolution time. Within our case study we find that there is indeed a spreading of the string in the angular direction, such that the string eventually covers the whole horizon. However, regarding the radial direction, we find that Lorentz-contraction suppresses the radial string spreading.

1 Introduction

Some of the most important problems in theoretical high energy physics today, are related to the quantum properties of matter near a black hole horizon (not to speak about the quantum properties of the black holes themselves). Especially the problem of the microscopic explanation of the black hole entropy and the problem of information loss, have attracted a lot of interest in recent years.

In many approaches to these problems, the so-called "stretched horizon" plays an important role. For an outside observer the stretched horizon appears as a membrane placed near the event horizon of the black hole, endowed with mechanical, electrical and thermal properties (see for instance [1], and references given therein). The interactions of the stretched horizon with the outside world can be thought of as arising from the boundary conditions that must be implemented. 't Hooft [2] and Susskind [3] further suggested that the stretched horizon exists also as a collection of quantum mechanical microscopic degrees of freedom which can absorb, store and re-emit any quantum mechanical information which falls into the black hole. The above description by an external observer is complementary [2, 3, 4] to the description provided by a freely falling observer, who crosses the horizon and ends up at the singularity. Within this spirit, Susskind [5] studied the spreading of a string approaching the event horizon of a black hole, as seen by a distant observer. Later is was argued by Mezhlumian et al [6], that string thermalization takes place, and eventually the information carried by the string is re-emitted as thermal Hawking radiation. To obtain these results, quantitative results from Rindler or Minkowski spacetime [7] were more or less directly taken over to the curved spacetime region near a black hole. For instance, in [6], the spherical horizon region is approximated by a plane, corresponding to the case of an infinite mass black hole.

The problem of the classical and quantum propagation of strings in a black hole background is rather complicated and generally not solvable. This is due to the highly non-linear nature of the equations of motion and the subsequent absence of a light-cone formulation (when using conformal gauge). Therefore, in the above mentioned papers a number of simplifying assumptions were made. Namely, it was assumed that the infalling string is very small in both the radial and angular directions, compared to the curvature radius of the black hole background. And then the horizon region of the black hole

spacetime was approximated first by "radial" Rindler space (with $R^2 \times S^{D-2}$ geometry) and eventually by "Cartesian" Rindler space (with flat R^D geometry). This allowed a light-cone formulation, and the equations of motion could be solved. However our ultimate goal, to study an extended string covering the whole horizon, renders the above assumptions and approximations inconsistent and inappropriate.

The purpose of the present paper is to address again this important issue, avoiding unwarranted approximations and carrying out all computations directly in the *exact* curved spacetime metric of a black hole. Everything will be expressed using Schwarzschild coordinates corresponding to a static asymptotic observer, in order to clarify what the static asymptotic observer actually sees. As already mentioned, there is no hope that we can solve the general string equations of motion in the exact curved Schwarzschild background. However, it is possible to find a special solution, and then more general solutions can be obtained by linearization of the equations of motion around this special solution. In this way we have complete control of the approximation involved.

Our physical setup will however be somewhat different from that discussed in references [5, 6]. In the original papers [5, 6], a small string was considered. The string was initially far outside the black hole, but then approached the horizon while oscillating around its center of mass. We will consider instead a macroscopic closed string initially encircling the black hole horizon. The string is oscillating around some average shape, which is a circle, but its overall average dynamics is a contraction towards the horizon. The string center of mass is therefore not part of the string, but is at all times located at the singularity of the black hole. For such string configurations it is possible to compute the string spreading in the radial and angular directions, while consistently working in the exact curved Schwarzschild background.

In the next section, we present the circular string solution in the Schwarzschild background and the perturbations around it. In section 3, we examine the string spreading, as the black hole horizon is approached, in both the angular and the radial directions. Finally in section 4, we present our conclusions.

2 Circular String Oscillations

Our starting point is the Polyakov action

$$S_{(0)} = \frac{-1}{4\pi\alpha'}\int d\tau d\sigma\,\sqrt{-h}\;h^{AB}G_{AB}, \tag{2.1}$$

where h_{AB} is the Polyakov metric while G_{AB} is the induced metric on the string world-sheet

$$G_{AB} = g_{\mu\nu}x^{\mu}_{,A}x^{\nu}_{,B}. \tag{2.2}$$

Here $(A, B) = (0, 1)$ are the world-sheet indices while $(\mu, \nu) = (0, 1, 2, 3)$ are the spacetime indices. For simplicity we consider strings in a 4-dimensional spacetime, but all our results are easily generalized to arbitrary dimensions.

The equations of motion corresponding to the action (2.1),

$$G_{AB} = \frac{1}{2}h_{AB}G^{C}{}_{C}, \tag{2.3}$$

$$\Box x^{\mu} + h^{AB}\Gamma^{\mu}_{\rho\sigma}x^{\rho}_{,A}x^{\sigma}_{,B} = 0, \tag{2.4}$$

contrary to the case of flat Minkowski space, cannot generally be solved in a curved spacetime with metric $g_{\mu\nu}$. In most cases it is however possible to find exact special solutions $(h_{AB},\ x^{\mu})$. In particular, in the Schwarzschild black hole spacetime a number of exact special solutions are known [8, 9, 10, 11, 12]. More general solutions can then be found by considering linearized perturbations $(\delta h_{AB},\ \delta x^{\mu})$ around such exact special solutions. Moreover, since we are interested only in physical (transverse) perturbations, δx^{μ} can be expanded on a set of unit-vectors normal to the world-sheet of the exact special solution

$$\delta x^{\mu} = n^{\mu}_{i}\Phi^{i}, \tag{2.5}$$

where

$$g_{\mu\nu}n^{\mu}_{i}n^{\nu}_{j} = \delta_{ij}, \qquad g_{\mu\nu}n^{\mu}_{i}x^{\nu}_{,A} = 0. \tag{2.6}$$

It is then straightforward to show that the fields Φ^{i} $(i = 1, 2)$ are governed by the action [13, 14, 15]

$$S_{(2)} = \frac{1}{2\pi\alpha'}\int d\tau d\sigma\,\sqrt{-h}\;\Phi^{i}\left[h^{AB}\mathcal{D}_{ikA}\mathcal{D}_{kjB} - \mathcal{V}_{ij}\right]\Phi^{j}, \tag{2.7}$$

where

$$\mathcal{D}_{ikA} = \delta_{ik}\nabla_{A} + \mu_{ikA}, \tag{2.8}$$

$$\mathcal{V}_{ij} = h^{AB} R_{\mu\rho\sigma\nu} x^{\mu}_{,A} x^{\nu}_{,B} n^{\rho}_{i} n^{\sigma}_{j} - \frac{2}{G^{C}{}_{C}} \Omega_{iAB} \Omega_{j}{}^{AB}. \tag{2.9}$$

Here we also introduced the extrinsic curvature and torsion of the world-sheet

$$\Omega_{iAB} = g_{\mu\nu} n^{\mu}_{i} x^{\rho}_{,A} \nabla_{\rho} x^{\nu}_{,B} \quad , \quad \mu_{ijA} = g_{\mu\nu} n^{\mu}_{i} x^{\rho}_{,A} \nabla_{\rho} n^{\nu}_{j}, \tag{2.10}$$

as well as the Riemann tensor $R_{\mu\rho\sigma\nu}$ of the curved spacetime.

In the present paper we take as the exact special (unperturbed) solution a circular string. That is, we take $h_{AB} = \eta_{AB}$ (gauge choice) and make the ansatz

$$x^0 = t = t(\tau), \quad x^1 = r = r(\tau), \quad x^2 = \theta = \pi/2, \quad x^3 = \phi = \sigma, \tag{2.11}$$

describing a circular string with time-dependent radius in the equatorial plane of the Schwarzschild black hole

$$ds^2 = -(1 - 2M/r) dt^2 + (1 - 2M/r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{2.12}$$

The equations of motion become

$$\dot{t} = \frac{E}{1 - 2M/r}, \tag{2.13}$$

$$\dot{r}^2 = E^2 - r^2(1 - 2M/r), \tag{2.14}$$

and they are solved by [9]

$$t(\tau) = E\tau + 2M \log\left|\frac{\tan(\tau/2) + \delta}{\tan(\tau/2) - \delta}\right|, \tag{2.15}$$

$$r(\tau) = M + \sqrt{M^2 + E^2}\ \cos(\tau), \tag{2.16}$$

where E is an integration constant and $\delta = (\sqrt{M^2 + E^2} - M)/E$. Physically this solution describes a macroscopic circular string, encircling the black hole in the equatorial plane, starting at $\tau = 0$ with maximal radius r_{max}

$$r_{max} = r(\tau = 0) = M + \sqrt{M^2 + E^2}. \tag{2.17}$$

It then contracts and crosses the horizon at finite world-sheet time τ_h

$$\tau_h = \tau(r = 2M) = \arccos(M/\sqrt{M^2 + E^2}\ ) \in\,]0, \pi/2[\ , \tag{2.18}$$

which of course corresponds to infinite coordinate time $t(\tau_h) = \infty$. The string eventually falls into the singularity at world-sheet time τ_0

$$\tau_0 = \tau(r = 0) = \arccos(-M/\sqrt{M^2 + E^2}\ ) \in\ ]\pi/2, \pi[\ . \tag{2.19}$$

The integration constant E has the physical interpretation of the constant energy of the string (in suitable units).

It is important to notice that the string at the horizon behaves as a null string [16] in the radial direction. For a null string the string tension is set to zero and the string appears as a collection of individual massless particles, each following its own null geodesic line. We suggest that this is a general feature: any kind of matter at the horizon looks like a collection of massless particles. This *horizon universality* is reminiscent of the universality encountered in the parton model (for early work on the parton model see [17]). In a hard collision involving a hadron (or hadrons) we probe the short distance structure of a hadron, and the partonic degrees of freedom, universal for all collisions, are revealed. Non-leading corrections to the parton model are sensitive to the specific character of the collision involved. In our case the black hole geometry provides the high energy required to probe the short distance behavior of the infalling matter, and it appears that any kind of matter at the event horizon of a black hole, that is in the ultraviolet limit, looks identical. The horizon universality is further reflected in the emitted thermal Hawking radiation [18, 19], which involves particles very close to the horizon and ignores the initial state of the infalling matter. We suspect that if we take into account the whole evolution of the infalling matter before reaching the horizon, then the emitted radiation will be non-thermal and will contain all the relevant information of the initial state (in close analogy to non-leading corrections to the parton model). The above emerging picture is in accordance with the line of reasoning advocated by 't Hooft [2]. Our concrete example also realizes the holographic principle [2, 20]: null rays connect the event horizon with the outside world, as well as with the interior of the black hole.

We now consider perturbations around the solution (2.15)-(2.16). The two physical polarizations of oscillations (2.5) are radial and angular oscillations, respectively, in the directions of the unit normal vectors

$$n_r^\mu = \left(\frac{\dot{r}}{r - 2M},\ \frac{\dot{t}}{r^2}(r - 2M),\ 0,\ 0 \right), \tag{2.20}$$

$$n^\mu_\theta = \left(0,\ 0,\ \frac{1}{r},\ 0\right). \tag{2.21}$$

After a little algebra, the action (2.7) becomes

$$S_{(2)} = \frac{1}{2\pi\alpha'}\int d\tau d\sigma\ \left\{\Phi_r\left(-\partial_\tau^2+\partial_\sigma^2+\frac{M}{r}+\frac{2E^2}{r^2}\right)\Phi_r+\Phi_\theta\left(-\partial_\tau^2+\partial_\sigma^2+\frac{M}{r}\right)\Phi_\theta\right\}. \tag{2.22}$$

When the string approaches the horizon, the action reduces to

$$S_{(2)} = \frac{1}{2\pi\alpha'}\int d\tau d\sigma\ \left\{\dot{\Phi}_r^2-\Phi_r'^2+\dot{\Phi}_\theta^2-\Phi_\theta'^2+\frac{1}{2}\left(1+\frac{E^2}{M^2}\right)\Phi_r^2+\frac{1}{2}\Phi_\theta^2\right\}. \tag{2.23}$$

The equations of motion corresponding to the action (2.23) are solved by

$$\Phi_r = \frac{\sqrt{\alpha'}}{2}\sum_{n>0}\frac{1}{\sqrt{n\Omega_n}}\left[a_n e^{-in(\Omega_n\tau-\sigma)}+\tilde{a}_n e^{-in(\Omega_n\tau+\sigma)}+c.c.\right], \tag{2.24}$$

$$\Phi_\theta = \frac{\sqrt{\alpha'}}{2}\sum_{n>0}\frac{1}{\sqrt{n\omega_n}}\left[b_n e^{-in(\omega_n\tau-\sigma)}+\tilde{b}_n e^{-in(\omega_n\tau+\sigma)}+c.c.\right], \tag{2.25}$$

where $(\Omega_n,\ \omega_n)$ are given by

$$\Omega_n = \sqrt{1-\frac{1+E^2/M^2}{2n^2}}\quad,\quad \omega_n = \sqrt{1-\frac{1}{2n^2}}. \tag{2.26}$$

Equations (2.24)-(2.25) together with eqs.(2.20)-(2.21) and eq.(2.5) thus provide the first order perturbations around the circular string (2.15)-(2.16), when the string is in the vicinity of the black hole horizon. And it should be stressed again that we are treating the Schwarzschild background *exactly*; no Rindler approximations are involved.

From the action (2.23) also follows that the momenta conjugate to $(\Phi_r,\ \Phi_\theta)$ are given by

$$\Pi_r = \frac{\delta S_{(2)}}{\delta\dot{\Phi}_r} = \frac{1}{\pi\alpha'}\dot{\Phi}_r\quad,\quad \Pi_\theta = \frac{\delta S_{(2)}}{\delta\dot{\Phi}_\theta} = \frac{1}{\pi\alpha'}\dot{\Phi}_\theta \tag{2.27}$$

with equal-time Poisson-brackets

$$\{\Pi_r,\ \Phi_r\} = \{\Pi_\theta,\ \Phi_\theta\} = -\delta(\sigma-\sigma'). \tag{2.28}$$

It follows that $(a_n, \tilde{a}_n, b_n, \tilde{b}_n)$ are properly normalized "oscillators"

$$\{a_n, a_m^*\} = \{\tilde{a}_n, \tilde{a}_m^*\} = \{b_n, b_m^*\} = \{\tilde{b}_n, \tilde{b}_m^*\} = -i\delta_{nm}, \tag{2.29}$$

that is, at the quantum level, the oscillators $(a_n, \tilde{a}_n, b_n, \tilde{b}_n)$ will become the standard creation and annihilation operators.

Notice also that the zero-modes were eliminated from the summations in eqs.(2.24)-(2.25). As discussed in reference [21], both the zero-modes as well as the $n = 1$-modes should in fact be eliminated since they do not represent "true" oscillations of a circular string; they merely describe rigid translations and rotations that do not change the shape of the string. Elimination of the zero-modes, in particular, ensures that the frequencies (2.26) are real for sufficiently small values of the integration constant E/M.

3 String Spreading

In this section we shall consider the spreading of the circular string, as seen by a static asymptotic observer. Up to first order perturbations around the circular string, we have

$$r(\tau, \sigma) = r(\tau) + \frac{E}{r(\tau)}\Phi_r(\tau, \sigma), \tag{3.1}$$

$$\theta(\tau, \sigma) = \frac{\pi}{2} + \frac{1}{r(\tau)}\Phi_\theta(\tau, \sigma), \tag{3.2}$$

where $r(\tau)$ is the unperturbed string (2.16). Then the radial and angular spreading due to the oscillations are

$$R_r \equiv r(\tau, \sigma) - r(\tau) = \frac{E}{r(\tau)}\Phi_r(\tau, \sigma), \tag{3.3}$$

$$R_\theta \equiv r(\tau, \sigma)\cos[\theta(\tau, \sigma)] = \Phi_\theta(\tau, \sigma). \tag{3.4}$$

Using eqs.(2.24)-(2.25), the (square of the) average spreading of a circular string near the horizon is then given by

$$< R_r^2 >= \frac{1}{2\pi}\int_0^{2\pi} d\sigma \; \frac{E^2}{4M^2}[\Phi_r(\tau, \sigma)]^2, \tag{3.5}$$

$$< R_\theta^2 >= \frac{1}{2\pi} \int_0^{2\pi} d\sigma \, [\Phi_\theta(\tau, \sigma)]^2, \tag{3.6}$$

which leads to

$$< R_r^2 >= \frac{E^2 \alpha'}{16M^2} \sum_{n>0} \frac{1}{n\Omega_n} \left[a_n a_n^* + \tilde{a}_n \tilde{a}_n^* + 2a_n \tilde{a}_n e^{-2in\Omega_n \tau} + c.c. \right], \tag{3.7}$$

$$< R_\theta^2 >= \frac{\alpha'}{4} \sum_{n>0} \frac{1}{n\omega_n} \left[b_n b_n^* + \tilde{b}_n \tilde{b}_n^* + 2b_n \tilde{b}_n e^{-2in\omega_n \tau} + c.c. \right]. \tag{3.8}$$

The constants $(a_n, b_n, \tilde{a}_n, \tilde{b}_n)$ are of the order 1, so that the summations in eqs.(3.7)-(3.8) formally diverge logaritmically. But we shall now argue that the infinite summations must be truncated for physical reasons.

Assume that the asymptotic observer watches the contracting oscillating string with a fixed finite resolution time ϵ measured in coordinate time t. Then he will be able to see frequencies ν fulfilling the inequality

$$\nu < \nu_t \equiv \frac{1}{\epsilon} \tag{3.9}$$

These frequencies correspond to frequencies ν_τ in world-sheet time τ

$$\nu_t \frac{dt}{d\tau} = \nu_\tau \quad \Leftrightarrow \quad \nu_\tau = E(1 - 2M/r)^{-1} \, \nu_t, \tag{3.10}$$

That is to say, as the string approaches the horizon, the asymptotic observer will see more and more oscillation modes, even though his resolution time is fixed and finite. This is of course nothing but the standard gravitational redshift effect.

However, we must also take into account the Lorentz-contraction of the radial oscillations. The Lorentz-contraction factor is

$$\gamma^{-1} = \sqrt{1 - v_p^2}\,, \tag{3.11}$$

where v_p is the physical speed of the contraction of the circular string

$$v_p = \frac{dl_{proper}}{d\tau_{proper}} = \left(1 - \frac{2M}{r}\right)^{-1} \frac{dr}{dt} = \left(1 - \frac{2M}{r}\right)^{-1} \frac{dr}{d\tau}\frac{d\tau}{dt} = \frac{1}{E}\sqrt{E^2 - r(r - 2M)}. \tag{3.12}$$

Such that

$$\gamma^{-1} = \frac{r}{E}\sqrt{1-2M/r}. \tag{3.13}$$

Thus altogether the asymptotic observer (AO) will see the following when the string approaches the horizon

$$< R_r^2 >_{AO} = \gamma^{-2}\frac{E^2\alpha'}{16M^2}\sum^{\nu_\tau}\frac{1}{n\Omega_n}\left[a_n a_n^* + \tilde{a}_n\tilde{a}_n^* + 2a_n\tilde{a}_n e^{-2in\Omega_n\tau} + c.c.\right], \tag{3.14}$$

$$< R_\theta^2 >_{AO} = \frac{\alpha'}{4}\sum^{\nu_\tau}\frac{1}{n\omega_n}\left[b_n b_n^* + \tilde{b}_n\tilde{b}_n^* + 2b_n\tilde{b}_n e^{-2in\omega_n\tau} + c.c.\right]. \tag{3.15}$$

That is to say,

$$|R_r|_{AO} \sim \left[\frac{\alpha'}{4}\left(1-\frac{2M}{r}\right)\log\left(\frac{E}{\epsilon}(1-2M/r)^{-1}\right)\right]^{1/2} \quad \rightarrow \quad 0 \ , \quad r \rightarrow 2M \tag{3.16}$$

$$|R_\theta|_{AO} \sim \left[\frac{\alpha'}{4}\log\left(\frac{E}{\epsilon}(1-2M/r)^{-1}\right)\right]^{1/2} \quad \rightarrow \quad \sqrt{\frac{\alpha'}{4}\left|\log(1-2M/r)\right|} \ , \quad r \rightarrow 2M \tag{3.17}$$

We therefore find that there is a string spreading in the angular direction, while Lorentz-contraction kills the corresponding radial spreading. Our asymptotic observer will see that the string, as it is approaching the distance $r = 2M$ (to be reached at infinite coordinate time), wraps around the (2-dimensional) horizon of the black hole. Thus, for an external observer, the information carried by the string is absorbed by the entire area of the event horizon.

4 Conclusion

One of the most intriguing subjects in physics is the behavior of matter nearby a black hole (usually exemplified by the problem of black hole entropy and the problem of information loss). It is not clear if the existing theoretical framework suffices to address the issue, or if a major revision of current concepts is necessary. One promising approach evolves around ideas proposed and elaborated by 't Hooft [2] and Susskind [3]. According to these ideas, all the information of infalling matter is stored in the stretched horizon

and is further re-emitted via a Hawking-type radiation. The external viewpoint and the comoving viewpoint are complementary to each other [2, 3, 4]. Along these lines Susskind [5] provided an analysis of the external viewpoint within string theory: how an infalling string into a black hole appears to an asymptotic observer. He concluded that the stringy degrees of freedom spread and cover the area of the event horizon.

In the present paper we studied the same problem in a different context, treating exactly the curved Schwarzschild background. We have considered a macroscopic oscillating string, encircling a Schwarzschild black hole, and its subsequent contraction towards the horizon. For such string configurations we were able to obtain analytic expressions for the angular and radial spreading, as seen by a static observer at spatial infinity. We took into account both the finite resolution time of the observer and the Lorentz-contraction factor.

We found a logarithmic growth of both the angular and radial string spreading. However the radial spreading is suppressed by the stronger Lorentz-contraction. On the other hand Susskind [5] and Mezhlumian et al. [6] obtained a radial spreading growing quadratically and persisting even after including the Lorentz-contraction effect. This difference might be due to the different physical configurations under study: for instance, the radial oscillations are in some sense "longitudinal" in the original situation [5, 6], while in our case they are actually "transverse". The difference might be related also to the simplifications employed in [5, 6] concerning the Rindler approximations. In both studies, however, the general picture emerges of a string covering eventually the whole black hole horizon.

Furthermore, our concrete physical example allowed us to establish the universal behavior of the infalling matter at the event horizon and lend further support to existing theoretical approaches to the information loss problem.

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